

CONVEXITIES IN SOME SPECIAL GRAPH CLASSES — NEW RESULTS IN AT-FREE GRAPHS AND BEYOND

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Abstract. We study convexity properties of graphs. In this paper we present a linear-time algorithm for the geodetic number in tree-cographs. Settling a 10-year-old conjecture, we prove that the Steiner number is at least the geodetic number in AT-free graphs. Computing a maximal and proper monophonic set in AT-free graphs is NP-complete. We present polynomial algorithms for the monophonic number in permutation graphs and the geodetic number in P_4 -sparse graphs.

1 Introduction

The geodetic number of a graph was introduced by Buckley, Harary and Quintas and by Harary, Loukakis and Tsouros [3,15] (see also [21] for a recent survey and [14] for pointing out some errors in [15]). It is defined as follows. A geodesic in a graph is a shortest path between two vertices, that is, a path that connects the two vertices with a minimal number of edges. Let $G = (V, E)$ be a graph. We write $n = |V|$ and $m = |E|$. For a set $S \subseteq V$ let

$$I(S) = \{ z \mid \exists_{x,y \in S} z \text{ lies on a } x, y\text{-geodesic} \}. \quad (1)$$

The set $I(S)$ is called the geodetic closure, or also, the interval of S . A set S is called convex if $I(S) = S$. A set S is geodetic if $I(S) = V$. The geodetic number $g(G)$ of G is defined as the minimal cardinality of a geodetic set.

For a set S , the interval $I(S)$ can be computed in $O(|S| \cdot m)$ time [8]. The computation of the geodetic number is NP-complete, even when restricted to chordal graphs, chordal bipartite graphs, or cobipartite graph [1,9,12]. It is

polynomial for cographs and splitgraphs [9], for unit interval graphs, bipartite permutation graphs and block-cactus graphs [12] and for Ptolemaic graphs [13]. It can be seen that the geodetic number problem can be formulated in monadic second-order logic (MSOL, see, eg, [18]).

Let $G = (V, E)$ be a graph and $W \subseteq V$. A *Steiner W-tree* is a connected subgraph of G with a minimal number of edges that contains all vertices of W . The Steiner distance of W is the size of a Steiner W -tree. The Steiner interval $S(W)$ is the set of all vertices that are in some Steiner W -tree. If $S(W) = V$ then W is a Steiner set. The Steiner number $s(G)$ is defined as the minimal cardinality of a Steiner set [21].

For graphs in general there is no order relation between the Steiner number and the geodetic number [16,21]. For distance-hereditary graphs it was shown that every Steiner set is geodetic, that is, $g(G) \leq s(G)$. The same was proved for interval graphs [16,21]. In their paper the authors posed the question whether the same holds true for AT-free graphs. We answer the question in Section 3.

The main result of this paper is on the geodetic number and the Steiner number. We start discussion on the geodetic number with a simple graph structure, the *tree-cograph*, defined in Section 2, and show that $g(G)$ can be computed in linear time when G is a tree-cograph. Next, in Section 3, we investigate the relation between the geodetic number and the Steiner number of an AT-free graph. We show that in an AT-free graph G , every Steiner set is geodesic, i.e., $g(G) \leq s(G)$. This answers a question posed by Hernando, Jiang, Mora, Pelayo, and Seara in 2005 [16]. A closely related concept, the *monophonic set* (defined in Section 4), is also investigated, and we show that computing a maximal and proper monophonic set is NP-complete, even for AT-free graphs.

Because of space limitations we relocate some results to appendices, including the 2-geodetic number for a tree-cograph (Appendix A), the geodetic number of a P_4 -sparse graph (Appendix B), and the *monophonic number*, which is the minimal cardinality of a monophonic set, of a permutation graph (Appendix C).

2 The geodetic number for tree-cographs

Tree-cographs were introduced by Tinhofer in [22]. They are defined as follows.

Definition 1. A graph G is a *tree-cograph* if one of the following holds.

1. G is disconnected.
2. \bar{G} is disconnected.
3. G or \bar{G} is a tree.

To compute the geodetic number for tree-cographs, we need an algorithm to compute the number for trees. In the following, we show how the computation can be done in linear time.

Lemma 1. *Let G be a graph and let x be a simplicial vertex. Then x is an element of every geodetic set in G .*

Proof. A simplicial occurs in a geodesic only as an endvertex. □

Lemma 2. *Let T be a tree. The set of leaves of T forms a minimum geodetic set.*

Proof. By Lemma 1, any minimum geodetic set contains all the leaves. Since any other, that is, internal vertex, of T lies on a geodesic between two leaves, the set of leaves forms a geodetic set. □

Lemma 3. *Let T be a tree with n vertices.*

- (a) *If $\text{diam}(T) \leq 2$ then $g(\bar{T}) = n$.*
- (b) *If $\text{diam}(T) = 3$ then $g(\bar{T}) = 2$.*
- (c) *If T has a vertex of degree two and $\text{diam}(T) > 3$ then $g(\bar{T}) = 3$.*
- (d) *Otherwise, $g(\bar{T}) = 4$.*

Proof. Assume that $n > 3$ and that $\text{diam}(T) \geq 3$. If $\text{diam}(T) = 3$ then $g(\bar{T}) = 2$. If T has a P_3 with the middle vertex of degree 2, then $g(\bar{T}) \leq 3$. In this case, if $\text{diam}(T) = 3$ then $g(\bar{T}) = 2$, and $g(\bar{T}) = 3$ otherwise.

Henceforth assume that T has no vertex of degree 2 and $\text{diam}(T) > 3$. It is easy to see that, if $\text{diam}(T) \geq 5$ then $g(\bar{T}) = 4$ and if $\text{diam}(T) = 4$ then a geodetic set needs all the vertices of a P_5 except some endpoint.

This proves the lemma. □

Instead of reducing to MSOL, we compute the geodetic number with a relatively simple and efficient algorithm. The proposed linear-time algorithm is based on a parameter called *2-geodetic number* of a graph, defined as follows.

Definition 2. *Let G be a graph. A geodetic set $S \subseteq V$ is 2-geodetic⁴ if every vertex x of $V \setminus S$ has two nonadjacent neighbors in S , that is, there are vertices $a, b \in N(x) \cap S$ such that $[a, x, b]$ induces a P_3 . The 2-geodetic number, $g_2(G)$, of G is the minimal cardinality of a 2-geodetic set in G .*

⁴ The 2-geodetic convexity should not be confused with the P_3 -convexity, studied, e.g., in [5].

Lemma 4. *There exists a linear-time algorithm to compute the 2-geodetic number for trees.*

Proof. The following is a dynamic programming algorithm to compute $g_2(T)$ for tree T rooted at an arbitrary vertex r . The subtree of T with root v is denoted by T_v and $C(v)$ denotes the set containing all children of v . Let S be a minimal 2-geodetic set. Let α_v , β_v , and γ_v be the numbers of vertices in $S \cap T_v$ when $v \in S$, $v \notin S$ and $|S \cap C(v)| = 1$, and $v \notin S$ and $|S \cap C(v)| \geq 2$, respectively. The values of α_v , β_v , and γ_v can be computed as follows.

$$\begin{aligned}\alpha_v &= \begin{cases} 1 & \text{if } v \text{ is a leaf} \\ 1 + \sum_{x \in C(v)} \min\{\alpha_x, \beta_x, \gamma_x\} & \text{otherwise.} \end{cases} \\ \beta_v &= \begin{cases} n & \text{if } v \text{ is a leaf} \\ \omega_{v,1} + \sum_{x \in C(v)} \min\{\beta_x, \gamma_x\} & \text{otherwise.} \end{cases} \\ \gamma_v &= \begin{cases} n & \text{if } v \text{ is a leaf} \\ \omega_{v,2} + \omega_{v,3} + \sum_{x \in C(v)} \min\{\alpha_x, \beta_x, \gamma_x\} & \text{otherwise.} \end{cases} \\ \text{Here } \omega_{v,1} &= \min_{x \in C(v)} \{\alpha_x - \min\{\beta_x, \gamma_x\}\}, \quad \omega_{v,2} = \min_{x \in C(v)} \{\alpha_x - \min\{\alpha_x, \beta_x, \gamma_x\}\}, \\ \omega_{v,3} &= \min_{x \in C(v) \setminus \{u\}} \{\alpha_x - \min\{\alpha_x, \beta_x, \gamma_x\}\}, \quad \text{and } u \text{ is a vertex with} \\ &\quad \omega_{v,2} = \alpha_u - \min\{\alpha_u, \beta_u, \gamma_u\}. \quad (2)\end{aligned}$$

Finally, we have $g_2(T) = \min\{\alpha_r, \gamma_r\}$. It is clear that $g_2(T)$ can be computed in linear time.

This completes the proof. \square

Lemma 5. *Assume T is a tree. Then*

$$g_2(\bar{T}) = \begin{cases} 3 & \text{if } \text{diam}(T) = 3 \text{ and there is a vertex of degree two} \\ 4 & \text{if } \text{diam}(T) = 3 \text{ and no vertex has degree two} \\ g(\bar{T}) & \text{otherwise.} \end{cases}$$

Proof. Checking the cases in the proof of Lemma 3 it follows that each of the geodetic sets in \bar{T} is actually a 2-geodetic set unless $\text{diam}(T) = 3$. \square

Remark 1. The 2-geodetic number of a tree-cograph can be computed in linear time (see Appendix A).

Based on this result, we have the following theorem.

Theorem 1. *There exists a linear-time algorithm that computes the geodetic number of tree-cographs.*

Proof. Assume that G is a tree-cograph. If G or \bar{G} is a tree then the claim follows from Lemmas 2 and 3.

Assume that G is disconnected. Let C_1, \dots, C_t be the components. Then

$$g(G) = \sum_{i=1}^t g(C_i),$$

where we write $g(C_i)$ instead of $g(G[C_i])$ for convenience.

Assume that \bar{G} is disconnected, and let C_1, \dots, C_t be the components of \bar{G} . Assume that the components are ordered, such that

$$|C_i| \geq 2 \quad \text{if and only if} \quad 1 \leq i \leq k.$$

We claim that

$$g(G) = \begin{cases} n & \text{if } k = 0 \\ g_2(C_1) & \text{if } k = 1 \\ \min \{ 4, g_2(C_i) \mid 1 \leq i \leq k \} & \text{if } k \geq 2. \end{cases} \quad (3)$$

To prove the claim, first observe that, when $k = 0$, G is a clique and $g(G) = n$. Assume that $k = 1$. Let D be a minimum 2-geodetic set in $G[C_1]$. Then it contains two nonadjacent vertices of C_1 . Then D is a geodetic set in G . Now let D' be a minimum geodetic set in G . It contains two nonadjacent vertices, which must be in C_1 . Then $D' \cap C_1$ is a 2-geodetic set in $G[C_1]$.

Assume that $k \geq 2$. Any 4 vertices, of which two are a nonadjacent pair in C_1 and another two are a nonadjacent pair in C_2 , form a geodetic set in G . Thus $g(G) \leq 4$. Assume that $g(G) = 2$. Then a minimum geodetic set must consist of two nonadjacent vertices, which must be contained in one C_i . Those form a 2-geodetic set in $G[C_i]$ and so Formula (3) holds true. Assume that $g(G) = 3$. It cannot be that two are a nonadjacent pair in one component and the third vertex is in another component, since then the two would be a geodetic set, contradicting the minimality. Therefore, the three must be contained in one component. This proves Formula (3).

This proves the theorem. □

3 Steiner sets in AT-free graphs

Asteroidal triples were introduced by Lekkerkerker and Boland to identify those chordal graphs that are interval graphs [20] (see also, eg, [19]).

An asteroidal triple, AT for short, is a set of 3 vertices $\{x, y, z\}$ such that there exists a path connecting any pair of them that avoids the closed neighborhood of the third.

A graph is AT-free if it has no asteroidal triple. Well-known examples of AT-free graphs are cocomparability graphs, that is, the complements of comparability graphs. However, AT-free graphs need not be perfect; the C_5 is AT-free.

Definition 3. A dominating pair in a connected graph is a pair of vertices such that every path between them is a dominating set.

The following result was proved in [2].

Lemma 6. Let G be an AT-free graph and let $W \subseteq V$. Let T be a Steiner W -tree. There exists a Steiner W -tree T' with $V(T) = V(T')$ and such that T' is a caterpillar.

Theorem 2. Let G be AT-free. Let W be a Steiner set of G . Then W is also a geodetic set. So, for AT-free graphs G holds that $g(G) \leq s(G)$.

Proof. Let $z \in V \setminus W$. We prove that there exist vertices $w', w'' \in W$ such that $z \in I(\{w', w''\})$.

By assumption, there exists a Steiner W -tree T that contains z , and by Lemma 6 we may assume that T is a caterpillar. Let P denote the backbone of the caterpillar, and let x and y be the endpoints of P . We may also assume that P is chordless, and x and y are leaves of T . Notice that all leaves of T are elements of W .

Since $z \in V \setminus W$, we have $z \in V(P)$. Let w' and w'' be two vertices on opposite sides of z such that

$$z \in P[w', w''] \quad \text{and} \quad P[w', w''] \cap W = \{w', w''\}.$$

If $P[w', w'']$ is not a shortest w', w'' -path in G , we claim that there is a cycle, which is the union of two paths Q and Q' in G , such that

- $Q = P[x', y']$
- $z \in V(Q)$
- Q' is a shortest x', y' -path in G .

- $V(Q') \cap V(P) = \{x', y'\}$
- $|V(Q')| < |V(Q)|$

Let P' be a shortest w', w'' -path in G , and let $V(P) \cap V(P') = \{a_1, a_2, \dots, a_k\}$, where elements are ordered according to the traversal of P' from w' to w'' . Clearly, $k \geq 2$, and for $i \in [k-1]$ we have that $P'[a_i, a_{i+1}]$ is a chordless path in G . Let i^* be the least integer such that $z \in V(P[a_{i^*}, a_{i^*+1}])$. It follows that $P'[a_{i^*}, a_{i^*+1}]$ is shortest and shorter than $P[a_{i^*}, a_{i^*+1}]$.⁵ By letting $x' = a_{i^*}$, $y' = a_{i^*+1}$, and $Q' = P'[x', y']$, we have the requested cycle.

In the following, we develop the case where $V(P[x', y']) \cap W \subseteq \{x', y'\}$. The case where $P[x', y']$ contains more vertices of W can be manipulated in a similar manner.⁶ Next, we claim that all vertices of Q and all leaves attached to these vertices, except those attached to only x_1 or y_1 , are adjacent to some vertex of Q' , where $N(x') \cap V(Q) = \{x_1\}$ and $N(y') \cap V(Q) = \{y_1\}$. Formally, let L be the set of leaves of T . The claim states that

$$\forall v \in V(Q) \cup L' \quad N[v] \cap V(Q') \neq \emptyset, \quad (4)$$

where $L' = \{v \in L \mid N(v) \cap (V(Q) \setminus \{x_1, y_1\}) \neq \emptyset\}$. To see that, assume that $u \in V(Q) \cup L'$ and $N(u) \cap V(Q') = \emptyset$. Then $\{x', y', u\}$ forms an asteroidal triple, since there is a u, x' -path that avoids $N[y']$ via Q , a u, y' -path that avoids $N[x']$ via Q and an x', y' -path that avoids $N[u]$ via Q' . Thus, condition (4) holds. Moreover, let q and q' be the length of Q and Q' , respectively. We may assume that $q \leq q' + 2$ since otherwise we can replace Q with Q' concatenated with x_1 and y_1 to get a smaller tree containing W .

Consider the following cases.

- (i) $| \{x', y'\} \cap \{x, y\} | = 0$: For each leaf u of T adjacent to only x_1 or y_1 , there is an asteroidal triple involving these endpoints, i.e. (x, u, y) , unless $N(u) \cap Q' \neq \emptyset$. Thus, with (4), we can replace Q with Q' to form the backbone of a caterpillar containing W . This contradicts the minimality of T .
- (ii) $| \{x', y'\} \cap \{x, y\} | = 1$: Assume that $x' = x$. Similar to the previous case, for each leaf u attached only at y_1 , we have (x, u, y) as an asteroidal triple, unless u is adjacent to some vertex of Q' . For $q' = q - 2$, or there is no leaf attached at x_1 , this leads to a contradiction, as in the previous case. Thus, we assume that there is a leaf w_1 attached only at x_1 and $q' = q - 1$. Notice that $N(w_1) \cap (V(P) \cup V(Q')) = \{x_1\}$. If $z = x_1$, then clearly z is on a shortest x, w_1 -path. If $z \neq x_1$, we claim that z is on a shortest w_1, w'' -path.

⁵ Otherwise, we can shorten P by replacing $P'[a_j, a_{j'}]$ by $P[a_j, a_{j'}]$, where $j \leq i^*$, $j' \geq i^* + 1$, and $P[a_j, a_{j'}]$ contains no a_i other than a_j and $a_{j'}$.

⁶ In this case, we choose Q as the maximal subpath of $P[x', y']$ containing no vertices in W except the endpoints, and Q' be the remainder of the cycle.

To see this, suppose to the contrary that

$$d(w_1, w'') < q + d(y', w''). \quad (5)$$

If there is a shortest w_1, w'' -path passing through a neighbor v of x , then

$$d(w_1, w'') \geq 2 + q' - 1 + d(y', w'') = q + d(y', w''),$$

since this path is of the form $w_1 \rightsquigarrow u \rightsquigarrow v \rightsquigarrow y \rightsquigarrow w''$, where u is a neighbor of w_1 . This contradicts (5), and we may assume that no shortest w_1, w'' -path contains a neighbor of x . However, this leads to the existence of an asteroidal triple (w_1, x, y) , again a contradiction. This proves the claim.

- (iii) $|\{x', y'\} \cap \{x, y\}| = 2$: We can shorten T by replacing the backbone if the leaves attached at neither x_1 nor y_1 . If $q' = q - 1$, or exactly one of x_1 and y_1 has a neighbor in $L \setminus L'$, then as in (ii), we have that z is on a shortest path between two vertices in W . The only case left of interest is when $q' = q - 2$, and there are leaves, w_1 and w_2 , attached at x_1 and y_1 , respectively. If $z \in \{x_1, y_1\}$, then z is on the shortest x, w_1 -path or y, w_2 -path. Otherwise, we claim that z is on a shortest w_1, w_2 -path.

To see this, suppose to the contrary that

$$d(w_1, w_2) < q. \quad (6)$$

If there is a shortest w_1, w_2 -path passing through two vertices which are neighbors of x and y , respectively, then we have

$$d(w_1, w_2) \geq 2 + q' - 2 + 2 = q,$$

since such a path is of the form $w_1 \rightsquigarrow u_1 \rightsquigarrow v_1 \rightsquigarrow v_2 \rightsquigarrow u_2 \rightsquigarrow w_2$, where $u_1 \in N(w_1)$, $v_1 \in N(x)$, $v_2 \in N(y)$, and $u_2 \in N(w_2)$. This contradicts (6). However, if each shortest w_1, w_2 -path contains no neighbor of x or no neighbor of y , then (x, w_1, y) or (y, w_2, x) is an asteroidal triple, again a contradiction.

This proves the theorem. \square

Although $g(G) \leq s(G)$ when G is AT-free, the equality is not guaranteed to hold even for subclasses like unit-interval graphs, as shown in Theorem 3.

Theorem 3. *For unit-interval graphs G , the geodetic number $g(G)$ and the Steiner number $s(G)$ are, in general, not equal. Moreover, the difference between the two numbers can be arbitrarily large.*

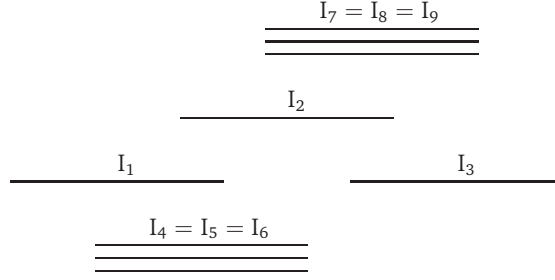


Fig. 1. An example showing that $g(G) \neq s(G)$ for unit-interval graphs in general. Our example graph G consists of nine unit intervals: $I_1 = [0.0, 1.0]$, $I_2 = [0.8, 1.8]$, $I_3 = [0.6, 2.6]$, $I_4 = I_5 = I_6 = [0.4, 1.4]$, $I_7 = I_8 = I_9 = [1.2, 2.2]$, and it follows that $g(G) \leq 4 < s(G)$.

Proof. Consider a set of seven unit-length intervals as depicted in Figure 1. Let G be the interval graph corresponding to these intervals; for ease of discussion, we abuse the notation slightly to refer to a vertex in G by the label of its corresponding interval. It is easy to check that $g(G) \leq 4$, as $\{I_1, I_3, I_4, I_7\}$ forms a geodetic set. We shall show that $s(G) > 4$, so that the first statement of the theorem follows.

First, I_1 and I_3 are simplicial vertices, so that any minimal Steiner set must include I_1 and I_3 . On the other hand, I_1 and I_3 alone do not form a Steiner set, since $\{I_1, I_2, I_3\}$ forms the only Steiner $\{I_1, I_3\}$ -tree and, eg, I_4 does not lie on that.

Next, if a Steiner set includes I_2 , it has to include all the remaining vertices (because the subgraph induced by any superset of $\{I_1, I_2, I_3\}$ is connected, and so, any remaining vertex would not be included in a Steiner tree). Similarly, if a Steiner set includes $\{I_1, I_3, I_4, I_7\}$ then it has to include all the remaining vertices.

Thus, either a Steiner set has size 9, or it must include I_4 but not I_7 (or vice versa); further, if a Steiner set includes I_4 , then it has to include I_5 and I_6 also. Thus, $s(G) \geq 5$. This completes the proof of the first statement. To show the second statement, it suffices to duplicate arbitrarily many disjoint (i.e., non-overlapping) copies of the set of intervals in our example. \square

4 Monophonic sets in AT-free graphs

Let $G = (V, E)$ be a graph. For a set $S \subseteq V$ the monophonic closure is the set

$$J(S) = \{ z \mid \exists_{x,y \in S} z \text{ lies on a chordless } x, y\text{-path} \}.$$

For pairs of vertices x and y we write $J(x, y)$ instead of $J(\{x, y\})$. A set S is monophonic if $J(S) = V$. The monophonic number $m(G)$ of G is the minimal

cardinality of a monophonic set in G . Some complexity results on monophonic convexity appear in [10]. Computing the monophonic number of a graph in general is NP-complete. In cographs, as in distance-hereditary graphs, chordless paths are geodesics, and so, $m(G) = g(G)$. The monophonic hull number turns out to be polynomial; actually, if a graph is not a clique and contains no clique separator, then it is called an atom, and in atoms, every pair of nonadjacent vertices forms a monophonic hull set [11] and [10, Theorem 5.1]

Theorem 4 (See [16]). *Let G be connected. Then every Steiner set in G is monophonic. Consequently,*

$$m(G) \leq s(G).$$

Remark 2. Let $c_m(G)$ denote the cardinality of a maximum, proper, monophonically convex subset of G .

Theorem 5. *Computing $c_m(G)$ is NP-complete for AT-free graphs.*

Proof. Computing the clique number ω is NP-complete for AT-free graphs [2]. Let H be AT-free. We may assume that $\omega(H) < |V(H)| - 1$. Let G be the graph obtained from H as follows. Add two nonadjacent vertices u and v and make each adjacent to all vertices of H . Notice that G is AT-free. As in [10], it is easy to see that $c_m(G) \geq k + 1$ if and only if $\omega(H) \geq k$. \square

Remark 3. Basically, an application of Dirac's theorem shows that the monophonic number of a chordal graph equals the number of simplicial vertices [7,13].

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A The 2-geodetic number for tree cographs

Lemma 7. *There exists a linear-time algorithm to compute the 2-geodetic number of tree-cographs.*

Proof. Let G be a tree-cograph. By Lemmas 4 and 5 we may assume that G is not a tree or the complement of a tree.

Assume that G is disconnected and let G be a union of two tree-cographs G_1 and G_2 . In that case

$$g_2(G) = g_2(G_1) + g_2(G_2).$$

Now assume that \bar{G} is disconnected. Let C_1, \dots, C_t be the components of \bar{G} and let them be ordered such that

$$|C_i| \geq 2 \quad \text{if and only if} \quad 1 \leq i \leq k.$$

If $k = 0$ then G is a clique and $g_2(G) = n$.

Assume that $k = 1$. Then

$$g_2(G) = g_2(C_1),$$

where we write $g_2(C_1)$ instead of $g_2(G[C_1])$, for convenience and clarity of notation. To see that, let D be a 2-geodetic set in G . Then $D \cap C_1$ is a 2-geodetic set in $G[C_1]$. Conversely, let D' be a 2-geodetic set in $G[C_1]$. Then D' contains two nonadjacent vertices since $G[C_1]$ is not a clique. Then D' is also a 2-geodetic set in G .

Assume that $k \geq 2$. We claim that

$$g_2(G) = \min \{ 4, g_2(C_i) \mid 1 \leq i \leq k \}.$$

Notice that a set of 4 vertices, 2 nonadjacent in C_1 and 2 nonadjacent in C_2 , form a 2-geodetic set in G . Thus $g_2(G) \leq 4$.

Assume that $g_2(G) = 2$. Then a minimum 2-geodetic set consists of two nonadjacent vertices, which must be contained in some C_i . Thus in that case,

$$g_2(G) = \min \{ g_2(C_i) \mid i \in \{1, \dots, k\} \}. \quad (7)$$

Assume that $g_2(G) = 3$. Assume that two vertices of a minimum geodetic set are in C_1 and one is in C_2 . Then the two in C_1 must be a 2-geodetic set, contradicting the minimality. Therefore, (7) holds also in this case.

This proves the lemma. □

B The geodetic number for P_4 -sparse graphs

Hoàng introduced P_4 -sparse graphs as follows.

Definition 4. A graph is P_4 -sparse if every set of 5 vertices induces at most one P_4 .

Jamison and Olariu characterized P_4 -sparse graphs using the notion of spiders.

Definition 5. A graph G is a thin spider if its vertices are partitioned into 3 sets, S , K and R , such that the following conditions hold.

1. S is an independent set and K is a clique and

$$|S| = |K| \geq 2.$$

2. Every vertex of R is adjacent to every vertex of K and to no vertex of S .
3. There is a bijection between K and S such that every vertex of S is adjacent to a unique vertex in K .

A thick spider is the complement of a thin spider.

Notice that, possibly $R = \emptyset$. The set R is called the head, K the body and S the set of feet of the spider. For a thick spider we switch the notation K and S for the feet and the body when taking the complement, so that in both cases the head R is adjacent to the body K .

The following characterization of P_4 -sparse graphs.

Theorem 6. A graph G is P_4 -sparse if and only if for every induced subgraph H of G one of the following conditions is satisfied.

- (a) H is disconnected.
- (b) \bar{H} is disconnected.
- (c) H is isomorphic to a spider.

Lemma 8. Let G be P_4 -sparse. There exists a linear-time algorithm to compute the 2-geodetic number $g_2(G)$.

Proof. The proof for the cases where G or \bar{G} is disconnected is similar to the proof of Lemma 7.

Assume that G is a thin spider, say with a head R , a body K and a set of feet S . Since all feet are pendant vertices, they all have to be in any minimum 2-geodetic set. So,

$$R = \emptyset \Rightarrow g_2(G) = |S| + 1,$$

since, choosing S and one vertex in K gives every other vertex of K two nonadjacent neighbors. When $R \neq \emptyset$, we have

$$g_2(G) = |S| + g_2(R),$$

where we write $g_2(R)$ instead of $g_2(G[R])$ for convenience. To see that, let D be a minimum 2-geodetic set in $G[R]$. Then $D \cup S$ is a 2-geodetic set in G , since every vertex of K is adjacent to one foot and one element of D . Now let D' be a minimum 2-geodetic set in G . Then $D' \cap R$ is a 2-geodetic set in $G[R]$, since for a vertex in R the two nonadjacent neighbors in D' must be in $D' \cap R$.

Assume that G is a thick spider. Let the head be represented by R . Let the body, which is a clique joined to R , be represented by K and let the set of feet be represented by S . Notice that,

$$R = \emptyset \quad \Rightarrow \quad g_2(G) = \begin{cases} 3 & \text{if } |K| = |S| = 2 \\ |S| & \text{otherwise.} \end{cases}$$

To see that, when $|K| = |S| = 2$ the graph is a P_4 , and $g_2(P_4) = 3$. When $|S| > 2$, all vertices of S must be in a minimum 2-geodetic set, since they are simplicials. Since $|S| > 2$, every vertex of K is adjacent to 2 nonadjacent feet.

Finally assume that G is a thick spider and assume that $R \neq \emptyset$. In that case,

$$g_2(G) = |S| + g_2(R).$$

The argument is similar to the one we gave above.

This proves the lemma. □

Theorem 7. *There exists a linear-time algorithm to compute the geodetic number of a P_4 -sparse graph.*

Proof. Let G be a P_4 -sparse graph. First assume that G is disconnected. Let C_1, \dots, C_t be the components of G . then

$$g(G) = \sum_{i=1}^t g(G[C_i]).$$

Assume that \bar{G} is disconnected. Let C_1, \dots, C_t be the components of \bar{G} . Assume that the components are ordered such that

$$|C_i| \geq 2 \quad \text{if and only if} \quad 1 \leq i \leq k.$$

Then we claim that

$$g(G) = \begin{cases} n & \text{if } k = 0 \\ g_2(G[C_1]) & \text{if } k = 1 \\ \min \{ 4, g_2(G[C_i]) \mid 1 \leq i \leq k \} & \text{if } k \geq 2. \end{cases} \quad (8)$$

To prove the claim, notice that, when $k = 0$, G is a clique and then $g(G) = n$. Assume that $k = 1$. Let D be a minimum 2-geodetic set in $G[C_1]$. Then this contains two nonadjacent vertices of C_1 . Then D is a geodetic set in G . For the converse, let D' be a minimum geodetic set in G . Then it contains two nonadjacent vertices, which must be in C_1 . Then $D' \cap C_1$ is a 2-geodetic set in $G[C_1]$, since any geodesic has length two. Assume that $k \geq 2$. Notice that 4 vertices, of which two are nonadjacent elements of C_2 and two are nonadjacent elements of C_2 , form a geodetic set. Therefore, $g(G) \leq 4$. Assume that $g(G) = 2$. Then a minimum geodetic set contains two nonadjacent vertices which must be in some component, say C_1 . Then those two vertices form a 2-geodetic set in $G[C_1]$, and Equation (8) holds true. Assume that $g(G) = 3$. It cannot be that two are in C_1 and one is in C_2 ; that is, all three must be in one component of \bar{G} . In that case, the three must form a 2-geodetic set in that component. This proves the claim.

Assume that G is a thin spider and let R , K and S be its head, body set of feet. By Lemma 1 all the feet are in any geodetic set. When $R = \emptyset$, then S is a geodetic set, since all vertices of K are in geodesics with endvertices in S . So we have

$$R = \emptyset \Rightarrow g(G) = |S|.$$

Assume that $R \neq \emptyset$. Then

$$g(G) = |S| + g_2(R),$$

where we write R instead of $G[R]$ for convenience. To see that, let D be a minimum 2-geodetic set in $G[R]$. Then $D \cup S$ is a geodetic set in G , since every vertex in K has two nonadjacent neighbors in the set, one in S and one in R . Let D' be a minimum geodetic set in G . Then $S \subseteq D'$. Furthermore, $D' \cap R$ is a 2-geodetic set in $G[R]$, since any geodesic has length 2 unless R is a clique.

Assume that $G[R]$ is a thick spider. Then

$$g(G) = |S| + g_2(R).$$

The analysis is similar to the treatment of the thin spiders.

This proves the theorem. □

C The monophonic number in permutation graphs

A permutation diagram consists of two horizontal lines L_1 and L_2 , one above the other. Each of the two lines has n distinct, designated points on it, labeled in an arbitrary order, $1, 2, \dots, n$. Each point on the topline is connected, via a straight line segment, to the point on the bottom line that has the same label.

The companion of a permutation diagram is a permutation graph. The n vertices of the graph are the n line segments in the diagram that connect the labeled points on the topline and bottom line. Any two vertices in the graph are adjacent precisely when the two line segments intersect each other. In general, a graph is a permutation graph if it is represented by a permutation diagram.

Notice that, if a graph G is a permutation graph then so is its complement \bar{G} . By the transitivity of the left-to-right ordering of parallel line segments, it follows that permutation graphs are comparability graphs. Baker et al. proved that these two properties characterize permutation graphs: a graph G is a permutation graph if and only if G and \bar{G} are comparability graphs (see, e.g., also [19]).

Lemma 9. *Permutation graphs are AT-free.*

Proof. Three, pairwise nonadjacent vertices in a permutation graph G , are represented by three parallel line segments in its diagram. Some line segment of a path that connects the outer two must intersect (or be equal to) the line segment that is in the middle of the three. This implies that the path hits the closed neighborhood of the vertex in the middle. \square

An elegant notion of betweenness for AT-free graphs was introduced by Broersma et al. as follows.

For two nonadjacent vertices x and y in a graph G denote by $C_x(y)$ the component of $G - N[x]$ that contains y .

Definition 6. *Let x and y be nonadjacent vertices. A vertex z is between x and y if*

$$z \in C_x(y) \cap C_y(x).$$

The following justification of this definition was proved in [2].

Theorem 8. *Let G be AT-free and let z be a vertex between nonadjacent vertices x and y . Then x and y are in different components of $G - N[z]$.*

Definition 7. A pair of vertices x and y is extremal if the number of vertices between them is maximal.

Lemma 10. Let G be AT-free and let $\{x, y\}$ be an extremal pair. Let

$$\Delta(x) = N(C_x(y)) \quad \text{and} \quad A(x) = V \setminus (A(x) \cup \Delta(x)).$$

Then every vertex of $A(x)$ is adjacent to every vertex of $\Delta(x)$.

Proof. Let C be the largest component of $G - N[x]$ and let $\Delta = N(C)$. Let $A = V \setminus (\Delta \cup C)$. Then every vertex of A is adjacent to every vertex of Δ , otherwise there would be a vertex x' for which the largest component of $G - N[x']$ properly contains C . \square

Notice that, by Theorem 8 and Lemma 10, every extremal pair is a dominating pair.

The following lemma gives the monophonic number for the graph induced by $A(x) \cup \Delta(x)$. For its proof we refer to [?].

Lemma 11. Let G_1 and G_2 be two graphs and let $H = G_1 \otimes G_2$, that is, H is the graph obtained from the union of G_1 and G_2 by adding all edges between pairs $x \in V(G_1)$ and $y \in V(G_2)$. Let $n_i = |V(G_i)|$, for $i \in \{1, 2\}$. Then

$$m(H) = \begin{cases} n_1 + n_2 & \text{if } G_1 \simeq K_{n_1} \text{ and } G_2 \simeq K_{n_2} \\ m(G_2) & \text{if } G_1 \simeq K_{n_1} \text{ and } G_2 \not\simeq K_{n_2} \\ m(G_1) & \text{if } G_1 \not\simeq K_{n_1} \text{ and } G_2 \simeq K_{n_2} \\ \min \{ 4, m(G_1), m(G_2) \} & \text{if } G_1 \not\simeq K_{n_1} \text{ and } G_2 \not\simeq K_{n_2}. \end{cases}$$

Let $\{x, y\}$ be an extremal pair. Then, by Lemma 10,

$$J(x, y) \subseteq \{x, y\} \cup \Delta(x) \cup \Delta(y) \cup (C_x(y) \cap C_y(x)).$$

Unfortunately, we don't always have equality.

Lemma 12. Let $C = C_x(y) \cap C_y(x)$ and let $z \in C$. Then

$$z \notin J(x, y) \Leftrightarrow \forall a \in \Delta_z(x) \forall b \in \Delta_z(y) \quad a = b \quad \text{or} \quad \{a, b\} \in E, \\ \text{where } \Delta_z(x) = N(C_z(x)) \quad \text{and} \quad \Delta_z(y) = N(C_z(y)). \quad (9)$$

Proof. When $z \in C$ and z is on a chordless x, y -path P , then z has neighbors u and v in P which are not adjacent. By Theorem 8, $N[z]$ separates x and y in different components, so $u \in N(C_z(x))$ and $v \in N(C_z(y))$ are not adjacent. For the converse, no chordless path can contain z when every vertex of $C_z(x)$ is equal or adjacent to every vertex in $C_z(y)$, since any such path would have a chord. \square

Two minimal separators S_1 and S_2 are parallel if all vertices of $S_1 \setminus S_2$ are contained in one component of $G - S_2$ and all vertices of $S_2 \setminus S_1$ are contained in one component of $G - S_1$. In such a case, let C_2 be the component of $G - S_1$ that contains $S_2 \setminus S_1$ and let C_1 be the component of $G - S_2$ that contains $S_1 \setminus S_2$. The vertices between S_1 and S_2 are the vertices of $C_1 \cap C_2$.

Minimal separators in permutation graphs were analyzed in [?] with the notion of a scanline. Consider a permutation diagram. A scanline is a line segment with one endpoint on the topline and one endpoint on the bottom line, such that neither of its endpoints coincides with a labeled point of the diagram. When S is a minimal separator in a permutation graph then there is a scanline s in the diagram such that the line segments that cross s are exactly the vertices of S .

Lemma 13. *Let S be a minimal separator in a permutation graph G and let C be a component in $G - S$. The neighborhoods in C of two nonadjacent vertices in S are ordered by inclusion.*

Proof. Consider a scanline s . All the line segments of a component occur on the same side of s ; say the left side. Let τ and ν be nonadjacent vertices whose line segments cross s . Say that τ and ν have their bottom endpoints on the left side of s and say that the endpoint of ν on the bottom line is closer to s than the endpoint of τ . Then every line segment of C that crosses ν also crosses τ , that is,

$$N(\nu) \cap C \subseteq N(\tau) \cap C.$$

This proves the lemma. □

Lemma 14. *Let $\{x, y\}$ be an extremal pair. Let Q be a minimum monophonic set. Then*

$$Q \cap A(x) = \emptyset \quad \Rightarrow \quad Q \cap \Delta(x) \neq \emptyset.$$

Proof. Let Q be a minimum monophonic set and assume that $Q \cap A(x) = \emptyset$. Assume that x is on a chordless q_1, q_2 -path P , for some q_1 and q_2 in Q . Then q_1 and q_2 are not adjacent. The chordless path P must contain two nonadjacent vertices q'_1 and q'_2 in $\Delta(x)$. But then, by Lemma 13 the path cannot be chordless, unless one of q_1 and q_2 is in $\Delta(x)$. □

The solutions for minimum monophonic sets Q with $x \notin Q$ and $y \notin Q$ are easily obtained by a modified input with some extremal pair in the solution. To simplify the description somewhat we henceforth assume that there is a minimum monophonic set which contains both elements of some extremal pair $\{x, y\}$.

Definition 8. *Let S_1 and S_2 be two minimal separators in a permutation graph G . Then S_1 and S_2 are successional if*

- (a) S_1 and S_2 are parallel, and
- (b) $a \in S_1$ and $b \in S_2$ implies $a = b$ or $\{a, b\} \in E$, and
- (c) The number of vertices between S_1 and S_2 is maximal with respect of the previous two conditions.

In the following we use the notation of lemma 10. Let $\{x, y\}$ be an extremal pair. Let C be a component of $G - J(x, y) - A(x) - A(y)$. Assume that the component C is between successional separators S_1 and S_2 . By definition of successional separators and by the nature of the permutation diagram, every vertex of C is adjacent to all vertices of S_1 or to all vertices of S_2 . In other words, if some vertex of S_1 has no neighbors in C then all vertices of S_2 are adjacent to all vertices of C . We say that a vertex $s \in S_1$ is partially adjacent to C if s has at least one neighbor and at least one nonneighbor in C .

We say that a component C is covered by a set of vertices Q , if every vertex of C is on a chordless path between vertices in Q . For a component C , let $\xi(C)$ denote the minimum number of vertices in G in any cover of C , that is,

$$\xi(C) = \min \{ |Q| \mid C \text{ is covered by } Q \}.$$

The algorithm recurses on components C that are between successional separators S_1 and S_2 and tabulates the covers of C , by distinct elements of S_1 and S_2 . When S_1 is joined to the component, and when it has two nonadjacent vertices, only covers of C with at most two elements of S_1 are of interest. When S_1 is a clique, joined to the component, then by Lemma 11, no vertex of S_1 is of interest for the cover. To prove that the recursion is polynomial, we show, in the following lemma, that a vertex of S_1 is *partially* adjacent to at most one component that is between S_1 and S_2 .

Lemma 15. *Each vertex of S_1 is partially adjacent to at most one component C between S_1 and S_2 .*

Proof. Let $s \in S_1$ and assume that s has a neighbor and a nonneighbor in C . Any other component C' has all line segments to the left or to the right of all line segments of C . Thus either s is not adjacent to any vertex of C' or to all vertices of C' . \square

Lemma 16. *Let C be a component between successional separators S_1 and S_2 . For any cover Q of C there exists an alternative cover Q' with $|Q'| \leq |Q|$ such that Q' has at most 4 vertices in S_1 .*

Proof. Let C be a component between successional separators S_1 and S_2 . Let Q be a cover of C . Assume that a vertex $c \in C$ is on a chordless path between $s \in Q \cap S_1$ and $c' \in Q \cap C$. Then the c', s -path can be extended from s to either x or y . Thus we may replace those vertices $s \in Q \cap S$ by the single vertex x or y in Q' . A similar argument applies to chordless paths between two elements of different components C and C' .

The other possibility is that a vertex $c \in C$ is on a chordless path between two vertices s_1 and s_2 in $Q \cap S$; that is, c is a common neighbor of s_1 and s_2 . Let C^* be the set of vertices in C that are common neighbors of pairs in $Q \cap S_1$. Let $c_1 \in C^*$ be the vertex whose line segment has an endpoint on the bottom line furthest from the scanline S_1 . Similarly, let $c_2 \in C^*$ be the vertex whose line segment has an endpoint on the topline furthest from the scanline S_1 . The two pairs of $Q \cap S_1$ that cover c_1 and c_2 cover all other vertices of C^* . \square

Theorem 9. *There exists a polynomial algorithm to compute the monophonic number of permutation graphs.*

Proof. We only sketch the detailed, but otherwise standard, dynamic programming algorithm.

The proposed algorithm builds a decomposition tree. The root of the tree represents the graph G , with an extremal pair $\{x, y\}$. The children are the components of $G - J(x, y) - A(x) - A(y)$. These components are recursively decomposed in subtrees.

By Lemmas 11 and 16 we may restrict to a polynomial number of monophonic covers of each component. By dynamic programming, the algorithm groups successive components together, and builds a table for the covers of intervals of components. Since the number of table entries is polynomial, the algorithm runs in polynomial time. \square

Remark 4. We conjecture that a similar algorithm works for AT-free graphs.